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Design of an Observer for Quantized Output Systems Using Orthogonal Projection

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This paper presents a state observer for linear systems with quantized outputs. The observer employs an orthogonal projection operation at quantizer output discontinuities to enhance its convergence rate for quantized output systems. Although there may be a significant quantization error on average, it is possible to design observers with an exponentially stable tracking error. We explain how to construct the orthogonal projection operation in a Hilbert space and prove the stability of the proposed observer by using the Lyapunov second method. In order to assess the value of the orthogonal projection operation in the proposed observer, the simple motor system with an optical encoder has been analyzed numerically.

Key Words : Observer, Quantized Outputs, Orthogonal Projection, Lyapunov Method

1. Introduction

Systems whose outputs take on discrete values are called quantized output systems. Systems with quantized outputs are common in the area of digital control where all information is represented using a quantized value rather than a continuous-level. In all digital plants, measured outputs are quantized prior to a control computation. In most cases, the quantization error is small compared with the system noise and is justifiably ignored. However, there are exceptions. For example, quantization is an issue in precision motion control systems employing motor-encoder pairs, in systems with limited switch outputs, or in any systems where continuous states are measured by digital means. On the other hand, careful modeling of quantization is not an issue in communication systems that operate at high noise levels. We show that the incorporation of knowledge of the quantization nonlinearity leads to an improvement in the state estimate with a minor increase in observer complexity. Much of the work on quantization in control systems seeks to simply bound the performance degradation caused by the use of a quantizer. Curry (1970) has developed maximum likelihood estimates for static linear systems driven by Gaussian noise and having quantized outputs. The extension to linear dynamic systems appears intractable analytically, however, Curry does derive approximate formulae for state estimates, work well with small quantizer steps. Another approach proposed by Schewppe (1968) propagates an ellipsoidal set which approximates the true system state by containment. This method only requires the knowledge of bounds on inputs, and bounds on output measurement (quantizer) error; the performance calculation is intractable analytically in this case as well. More recently, Miller, Michel, and Farrel (1989) established useful bounds on tracking performance in digitally controlled plants, where there are numerical quantization in the digital computation of the control input. Delchamp, in (Delchamps, 1988, 1989a, 1989b), takes a new and inovative approach to dealing with quantization. Rather than treating quantization as a bounded disturbance, his method treats quantization exactly in the linear dynamic case, and establishes conditions under which the uncertainty in the system state tends to zero (as measured by

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differential entropy). The approach uses the system input to optimize the information acquired on the state, but mixes approaches aimed at addressing information. Therefore, simultaneous performance tracking now appear as a possibility. More recent works by Rotea and Williamson (1994) is representative of a broad class of problems focused on choosing state-space realization of discrete-time linear time-invariant systems which perform well in computer implementations. These methods effectivly treat numerical round-off quantization as a noise source and address the scaling of internal signals to optimize competing objectives of a) low sensitivity to quantization and b) the desire for infrequent numerical overflow. The approach contrasts with ours where quantization is modeled as a nonlinearity, rather than a noise source. More recent work addressing chaos in feed-back systems with quantization is done by Steppan and Haller (1996). Previous works are fundamental to the feedback problem in discrete time systems; whereas, we focus on the continuous time problem and observers.

In contrast to the aforementioned approaches, we attempt, in this paper, an exact analysis of the quantization. Quantization is a deterministic nonlinearity as in (Sur and Paden, 1996, 1997) and, by treating it as such, we obtain excellent state tracking performance in an observer. This exact treatment of quantization nonlinearity is similar in spirit to the work of Delchamps (1988, 1989a, 1989b).

This paper has the following format: in Sec. 2, we motivate and introduce the observer for the systems with quantized outputs. In Sec. 3, we modify existing Lyapunov theory to accommodate the discontinuous updates used in our observer, and prove the error convergence for the case of stable and unstable plants. Sec. 4 contains simulation results for the stable and unstable plant cases. Our conclusions are made in Sec. 5.

2. Problem Statement

We consider the observer design problem for a SISO linear time-invariant system with quantized

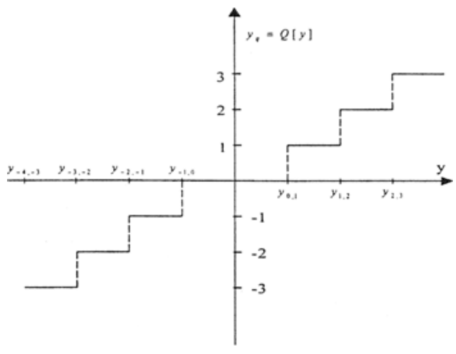

Fig. 1 Round-off quantization nonlinearity.

outputs.

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \tag{1}$$
$$y = Cx \tag{2}$$
$$y_q = Q[y] \tag{3}$$

We have taken the feedthough term D=0 for simplicity although it is straightforward to incorporate. The quantizer nonlinearity Q[·] in Eq. (3) is taken to be the round-off nonlinearity of Fig. 1. A typical approach to designing an observer for the system above is to simply ignore the quantizer. For example, a Luenberger observer has the form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y_q - \hat{y}), \quad \hat{x}(t_0) = \hat{x}_0 \tag{4}$$
$$\hat{y} = C\hat{x} \tag{5}$$

The state estimation error dynamics of the linear observer with quantizer is given by

$$\begin{aligned} \dot{e} &= Ae - L(y_q - \hat{y}), \quad e(t_0) = e_0 \\ &= Ae - L(y + \Delta y_q - \hat{y}) \\ &= (A - LC)e - L\Delta y_q \end{aligned} \tag{6}$$

where $e = x - \hat{x}$, $\Delta y_q = y_q - y$, and L is designed so that $(A - LC)$ is Hurwitz. Thus there exists an $M = M^T > 0$, $Q = Q^T > 0$ such that $(A - LC)^T M + M(A - LC) = -Q$. $V(t, e) = e^T M e$ is a Lyapunov function for the standard Luenberger observer. The time derivative of the Lyapunov function is

$$\frac{d}{dt} V(t, e) = 2\, e^T M L \Delta y_q - e^T Q e \tag{7}$$

Since quantizer with unit width has quantization error as $|\Delta y_q| \le 1/2 < \infty$, then for $\|e\|$ sufficiently large the sign of $\dot{V}$ will be dominated by the sign of the term $-e^T Q e$ which has been chosen to be negative definite. Therefore, there exists a region $\|e\| \leq \rho < \infty$ which is the complement of a compact sets of points enclosing the origin, within which $\dot{V}$ is negative and therefore, the solution of system (6) is uniformly bounded. The estimate of the domain of uniform boundedness for the solution $e(t)$ of system (6) is readily obtained from Eq. (7), where $|\Delta y_q|_{\max} \leq 1/2$, as follow: The worst contribution to the error of estimation, when the quantization Δy_q has the maximum value, $|\Delta y_q|_{\max} \leq 1/2$, and has a sign such that the term $2e^T ML\Delta y_q$ in Eq. (7) is positive. The requirement on e such that $\dot{V}$ should remain negative under this condition is obtained from the Eq. (7) as

$$|e^T ML| \leq e^T Qe \tag{8}$$

However, since Q is a real, symmetric positive definite matrix, it is known that $e^T Qe \geq \min \lambda_i (Q) \|e\|^2$ for $i = 1, \cdots, n$: thus the inequality of Eq. (8) is implied by

$$|e^T ML| \geq \min \lambda_i (Q) \|e\|^2 \tag{9}$$

A simplified more conservative expression which assumes the inequality of (9) is obtained by the use of a basic inequality involving the norms of matrices, as

$$\|ML\| \; \|e\| \geq \min \lambda_i (Q) \|e\|^2 \tag{10}$$

It is evident that if the equality sign is used in Eq. (10), the resulting expression for $\|e\|_{\max} = \rho$ is upper bounded on the domain of uniform boundness defined as

$$\min \lambda_i (Q) \rho^2 - \|ML\| \rho = 0 \tag{11}$$

Equation (11) has two real solution for ρ, only one of which satisfy $\rho = \frac{\|ML\|}{\min \lambda_i (Q)}$. Therefore, there is no guarantee that the error of state estimation goes to zero as $t \rightarrow \infty$ if there exist quantized error. The trajectories of most points are periodic and dense in the ball $B_\rho = \{e \in B_\rho: \dot{V}(e, t) > 0, e \notin B_\rho: \dot{V}(e, t) \leq 0\}$.

3. The Projection Algorithm

Herein a different algorithm is introduced for designing an observer which can eliminate an estimation error asymptotically. In our observer design, the discontinuities play a critical role since the only cases that y is measured exactly is at discontinuities. At other case, y_q differs from y up to 1/2. To see how the quantizer discontinuities can be used to estimate the state $\hat{x}$, consider Fig. 2. In this figure the state space is represented as the direct sum of range of C^T and the nullspace of C. A full rank matrix Γ is chosen such that $R(\Gamma) = N(C)$. In addition, let M satisfy the Lyapunov equation $A^T M + MA = -I$, and define a natural inner product on the state space by $\langle x, y \rangle_M = x^T My$. In this Hillbert space, the orthogonal projection matrix onto $N(C)$ is given by $P = C(C^T MC)^{-1} C^T M$.

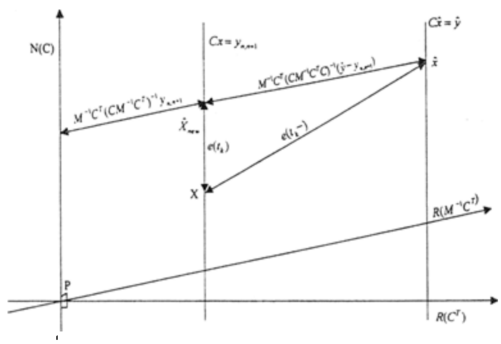

Fig. 2 The projection step in the observer.

Lemma 1 Let $M = M^T > 0$. Then there exists a vector C such that $N(C)^{\perp M} = R(M^{-1} C^T)$.

Proof : Let $N(C) = \{x \in R^n | Cx = 0\}$, then there exists a $N(C)^{\perp M} = \{y \in N(C)^{\perp S} | \langle x, y \rangle_S = 0, \forall x \in N(C)\}$. Let $y \in R(M^{-1} C^T)$, then there exists a $z \in R^n$ such that

$$\begin{aligned} y &= M^{-1} C^T z \\ y^T Mx &= z^T C (M^T)^{-1} Mx \\ &= z^T Cx \\ &= 0 \end{aligned}$$

Thus, $y \in N(C)^{\perp M}$. Let $y \in N(C)^{\perp M}$, then we have

$$\begin{aligned} y^T Mx &= 0, \; \forall \; x \in N(C) \\ &= Cx \\ &= z^T CM^{-T} Mx \\ y^T M &= z^T CM^{-T} M \\ y &= M^{-1} C^T z \end{aligned}$$

Thus, $y \in R(M^{-1} C^T)$. Therefore, there exists a vector C such that $N(C)^{\perp M}=R(M^{-1}C^T)$ ■

The corresponding projection onto $\langle\cdot, \cdot\rangle_M$, $N(C)^{\perp M}=R(M^{-1}C)$ is I-P. At a quantizer transition, the value of y is known exactly to be, say, y_q. As a consequence, the state satisfies the equation $Cx=y_q$ and lies on the hyperplane as shown in Fig. 2. If $\hat{x}$ is the present estimate of the state, the estimate can be improved by projecting it along $R(M^{-1}C)$ to the nearest point (with respect to $\langle\cdot, \cdot\rangle_M$) in the hyperplane containing x. This is the basic projection step used with our observer. The projection can be calculated in terms of $y_q=Q[y]$ to be

$$\hat{x}_{update} \leftarrow M^{-1}C(C^TM^{-1}C)^{-1}(\hat{y}-y_q) \tag{12}$$

Let t_k be the time at which a quantizer transition occurs, and let y_q be the corresponding transition value. We define our observer by

$$\dot{\hat{x}}=A\hat{x}+Bu, \ \forall \ t\notin[t_k]_{k=0}^{\infty}$$
$$\hat{x}_{update} \leftarrow \hat{x}-M^{-1}C(C^TM^{-1}C)^{-1}(\hat{y}-y_q), \tag{13}$$

where the arrow "←" indicates a discrete update. Next we will extend our methodology of the design state observer for a MIMO linear system with quantized outputs. The equation of the observer based on the projection algorithm can be expressed by

$$\dot{\hat{x}}=A\hat{x}+Bu, \ \forall \ t\notin[t_k]_{k=0}^{\infty}$$
$$\hat{x}_{update} \leftarrow \hat{x}-M^{-1}C_i(C_i^TM^{-1}C_i)^{-1}(\hat{y}_i-y_{qi}),$$
$$\forall t\in[t_k]_{k=0}^{\infty}, \ \forall i=1,2, \cdots, q, \tag{14}$$

where $C\in R^{n\times q}$, $i=1, 2, \cdots, q$ is output numbers, and M can be computed from $A^TM+MA=-I$. The case of a multi-output is to increase the information of the outputs and the number of projections. Intuitively, the estimation error of a state observer based on the multiple outputs should have an improved convergence rate relative to a single output system. In Eq. (14), t_{ki} is the time of the kth transition to the ith output. When multiple quantizer transitions occur at the same time, the respective projections are executed in arbitrary order. More sophisticated schemes can be derived for simultaneous transitions. However, for practical families of state trajectories with associated probability measures, the probability of simultaneous transitions is zero.

4. Stability Analysis

In this section, we present an extension of Lyapunov's second method for the investigation of the stability of linear time-invariant systems with resetting described by the projection algorithm in the previous section. We first define a solution to a differential equation with resetting at time t_k. Consider the differential equation with resetting

$$\dot{x}=f(x, t)$$
$$x(t_k) \leftarrow x_k, \ \forall k=0, 1, \cdots, n, \tag{15}$$

where $n\in[R, \infty]$, and there are finite t_k in any finite interval. Define x(t) be the caratheodory solution to $\dot{x}=f(x, t)$, $x(t_k)=x_k$ on the interval $[t_k, t_{k+1})$. Then x(t) is defined on all of $[t_0, \infty]$, provided $f(x, t)$ satisfies the assumption of piecewise continuity in t and local Lipschitz condition (Khalil, 1992). Define $x(t_{k^-})=\lim_{t\to t_k} x(t)$. Lyapunov stability theory for systems of the form (4) is easily generalized from standard results. All that is required to deal with the resetting, is to require that a Lyapunov function not increase upon a reset. As an illustration, we generalize theorem 4.1 of (Khalil, 1992). Let $f(x, t)$ be Lipschitz continuous on a domain $D=\{x\in R|\ \|x\|<r\}$, then we have the following.

Theorem 1 *Consider system (4) where t_k is a finite or countable set of resetting times. Let $x=0$ be an equilibrium point of $x=f(x, t)$, and $D=\{x\in R|\ \|x\|<r\}$. Let $V: [0, \infty)\times D\rightarrow R$ be a continuously differential function such that*

$$\alpha_1(\|V(t, x)\|)\leq V(t, x)\leq\alpha_2(\|x\|) \tag{16}$$
$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(x, t)\leq-\alpha_3(\|x\|), \ \forall t\notin\{t_k\} \tag{17}$$
$$V(x(t_k, t_k))-V(x(t_{k^-}), t_k)\leq 0, \ \forall t\in\{t_k\},$$
$$\forall x\in D, \tag{18}$$

where α_1, α_2, and α_3 are class K function defined on $[0, r)$. Then $x=0$ is uniformly asymptotically stable.

Proof : See the reference (Khalil, 1992). The only change in the proof required is that (18) in addition (17) is required to show that V is decreasing. ■

Corollary 1 *If all of the assumptions of the theorem 1 are satisfied with $\alpha_i(r) = k_i r^c$, for some positive constants k and c, then $x = 0$ is exponentially stable. Moreover, if the assumption holds globally, then $x = 0$ is globally exponentially stable.*

Note that we have not formally defined the various kinds of stability for the differential equation with resetting, but the generalizations are so slight that this is not required. Return now to the combined plant observer systems :

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ y_q &= Q[y], \qquad (19) \\ \dot{\hat{x}} &= A\hat{x} + Bu \\ \hat{y} &= C\hat{x} \\ \hat{x}_{update} &\leftarrow \hat{x} - M^{-1}C(C^T M^{-1} C)^{-1}(\hat{y} - y_q). \qquad (20) \end{aligned}$$

Denote the state error $e(t) = x(t) - \hat{x}(t)$, then we have the estimated error dynamics with discontinuity at quantization time t_k as follows:

$$\begin{aligned} \dot{e} &= Ae,\ e(t_0) = e_0,\ \forall t \in \{t_k\}_{k=0}^{\infty} \\ e_{update} &\leftarrow (I - P)e,\ \forall t \in \{t_k\}_{k=0}^{\infty}, \qquad (21) \end{aligned}$$

where $P = M^{-1}C^T(CM^{-1}C^T)^{-1}C$ is the projection matrix. This error is characterized by the fact that at times t_k of quantizer transition it experiences a sudden change and the estimated error dynamics can be rewritten as

$$\dot{e} = Ae - \textstyle\sum_{k=1}^{\infty} P\delta(t - t_k)e,\ \forall k = 1, 2, \cdots. \qquad (22)$$

The solution of an error dynamics in Eq. (21) or (22) can be expressed by

$$\begin{aligned} e(t, t_0, e_0) &= a^{A(t-t_k)} \textstyle\prod_{i=1}^{k} (I - P) e^{A\Delta t_k} e_0, \\ \forall t &\geq t_0. \qquad (23) \end{aligned}$$

Theorem 2 *Suppose that there exists $M^T = M > 0$ such that A satisfies $A^T M + MA = -I$. Then the origin of the system is exponentially stable.*

Proof Let $V(t, e) = e^T Me$ be a Lyapunov function of system (21) or (22). Then for $V(t, e) \in D$, we have $\dot{V}(t, e) = -e^T e = -\|e\|^2$. For $V(t, e) \in \Omega_k$, where Ω_k is a discontinuous set at quantized transition, we have $V(t_k, e) - V(t_k^-, e) = -(\hat{y}(t_k) - y_q(t_k))^T (C^T M^{-1} C)^{-1} (\hat{y}(t_k) - y_q(t_k))) \leq 0$. It follows that the estimated error goes to zero exponentially. ■

This proves the stability of the observer for the linear time-invariant stable system with quantized outputs. In the preceding theorem, the Lyapunov function is monotonically decreasing along the solutions of the system and is forced down by discrete updates to converge to zero more quickly as t tends to infinity. However, the rate of convergence is hard to define since we can not quantify the rate at which quantizer transitions occur. In this case, the monotone property of Lyapunov functions along solutions of the system (23) no longer holds for unstable systems. We need another condition for stability given in the following theorem.

Theorem 3 *The observer based on the projection algorithm (20) is exponentially stable if there exist $P \in R^{n\times n}$, $\lambda > 0$, and $\Delta T > 0$ such that*

$$\begin{aligned} &e^{A^T\Delta t_k}(I - P)^T M (I - P) e^{A\Delta t_k} - e^{-\lambda\Delta t_k} M < 0, \\ &\forall \Delta t_k \leq \Delta T \qquad (24) \end{aligned}$$

where $P = M^{-1}C(C^T M^{-1} C)^{-1} C^T \in R^{n\times n}$, and Δt_k are finite quantized time intervals.

Proof From the estimated error equation of (20), the solution can be obtained by

$$e(t) = e^{A\alpha} \textstyle\prod_{k=0}^{n} (I - P) e^{A\Delta t_k} e_0$$

where $t = \alpha + \sum_{k=1}^{n} \Delta t_k$, $0 \leq \alpha \leq \Delta t_{k+1}$ and $\Delta t_k = t_{k+1} - t_k$. Let $V(k, e) = e^T(k) Me(k)$ be a Lyapunov function of system (20). Then the Lyapunov function at italic font can be expressed by

$$\begin{aligned} V(k+1, e) &= e(k+1)^T Me(k+1) \\ &= e(k)^T e^{A^T\Delta t_k}(I - P) M (I - P) e^{A\Delta t_k} e(t_k). \qquad (25) \end{aligned}$$

Suppose that $e^{A^T\Delta t_k}(I - P)^T M (I - P) e^{A\Delta t_k} - e^{-\lambda\Delta t_k} M < 0$. Then we have

$$\begin{aligned} &e^{A^T\Delta t_k}(I - P)^T M (I - P) e^{A\Delta t_k} < e^{-\lambda\Delta t_k} M \\ &e(k)^T e^{A^T\Delta t_k}(I - P)^T M (I - P) e^{A\Delta t_k} e(k) \\ &< e^{-\lambda\Delta t_k} e(k)^T Me(k) \\ &V(k+1, e) < e^{-\lambda\Delta t_k} V(k, e) \\ &\frac{V(k+1, e)}{V(k, e)} < e^{-\lambda\Delta t_k}. \qquad (26) \end{aligned}$$

Because $\lim_{k\to\infty} V(k, e)$ converges to zero exponentially, e(t) converges to zero exponentially as $t \to \infty$. ■

5. Application to Motor Control

To assess the value of the projection operation in our observer we consider the simple DC-motor system with an optical encoder as shown in Fig. 3.

We simulate the simple motor optical encoder system modeled by

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix}=\begin{bmatrix}\chi & 0\\ 1 & -\frac{\alpha}{\beta}\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix}+\begin{bmatrix}\gamma\\ 0\end{bmatrix}u$$

$$y_q=Q[\theta] \tag{27}$$

where θ is angular position, ω is angular velocity, and χ, α, β are known coefficients of the DC-motor (Friendland, 1986). The quantized output from the encoder is $y_q=Q[\theta(t_k)]$, and is used to calculate $[\hat{\theta}(t_k), \hat{\omega}(t_k)]^T$ in order to reset the designed motor observer using results of Sec. 2. The simulation results of the motor system are shown in Figs. 4, 5 and 6. The estimated error of the closed-loop system with the projection operation decays faster than without the projection operation for a stable system case as shown in Fig. 4. We also simulated an quantized motor system using the projection operation and linear observer. Figure 5 shows the case where the quantizer and linear observer were used without our methodology. The error of the state estimation due to the quantization error is shown in Fig. 5. As shown in Fig. 6, the estimated angular position and velocity of the closed-loop system with an observer based upon the projection operation converge to the reference angular position and velocity. This result shows that an observer of an quantized motor system can be designed by using the projection operation even if it has quantized outputs. The projection approach for designing the observer is easy and profitable for systems with quantized outputs.

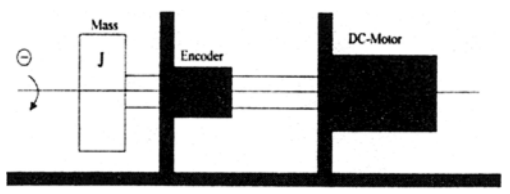

Fig. 3 DC-motor driving inertia load.

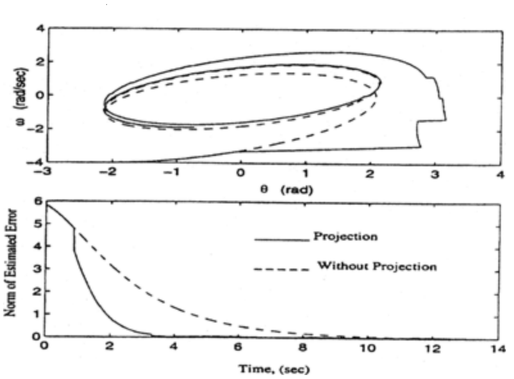

Fig. 4 The state trajectories and estimated error of stable DC-motor with and without projection and operation.

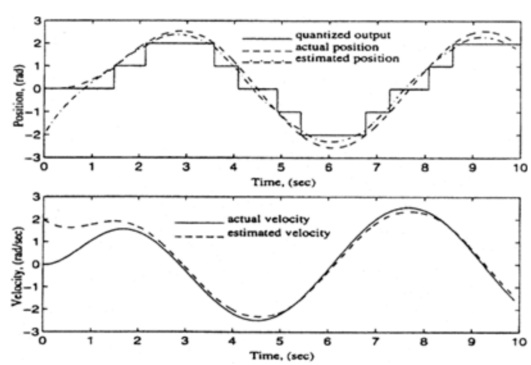

Fig. 5 Simulated closed-loop states of DC-motor with the quantizer and a linear observer.

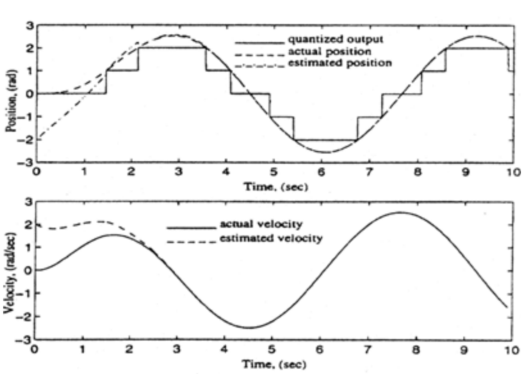

Fig. 6 Simulated closed-loop states DC-motor with an observer based upon the projection algorithm.

6. Conclusion

The objective of this paper is to demonstrate that a quantized measurement may be viewed profitably as limited information to extract the estimated state using the projection algorithm. The quantized output measurement can be taken into account explicity for the observer design. The observer for the quantized output system has an advantage that the estimated error approaches zero as $t \rightarrow \infty$ in spite of the limited output information as shown in Theorems 1, 2 and via simulation results of the stable DC-motor example. For quantized output systems, if the necessary and sufficient condition for the observer design (22) is satisfied, then the estimated error also goes to zero as shown in figure 6. From these simulation results, we may conclude that the observer based on the projection algorithm offers many advantages with minor increase in complexity.

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